

SU(2) reductions in $\mathcal{N}=4$ multidimensional supersymmetric mechanics

Stefano Bellucci^a, Sergey Krivonos^b and Anton Sutulin^b

^a *INFN-Laboratori Nazionali di Frascati, Via E. Fermi 40, 00044 Frascati, Italy*

^b *Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia*

Abstract

We perform an $su(2)$ Hamiltonian reduction in the bosonic sector of the $su(2)$ -invariant action for two free $(4, 4, 0)$ supermultiplets. As a result, we get the five dimensional $\mathcal{N}=4$ supersymmetric mechanics describing the motion of an isospin carrying particle interacting with a Yang monopole. We provide the Lagrangian and Hamiltonian descriptions of this system. Some possible generalizations of the action to the cases of systems with a more general bosonic action, a four-dimensional system which still includes eight fermionic components, and a variant of five-dimensional $\mathcal{N}=4$ mechanics constructed with the help of the ordinary and twisted $\mathcal{N}=4$ hypermultiplets were also considered.

1 Introduction

The supersymmetric mechanics describing the motion of an isospin particle in the background non-Abelian gauge fields attracted a lot of attention in the last few years [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], especially due to its close relation with higher dimensional Hall effects and their extensions [11], as well as with the supersymmetric versions of various Hops maps (see e.g. [1]). The key point of any possible construction is to find proper “room” for semi-dynamical “isospin” variables, which have to be invented for the description of monopole-type interactions in Lagrangian mechanics. In supersymmetric systems these isospin variables should belong to some supermultiplet, and the main question is what to do with additional fermions accompanying the isospin variables. In [3] the additional fermions, together with isospin variables, span an auxiliary $(4, 4, 0)$ multiplet with a Wess-Zumino type action possessing an extra $U(1)$ gauge symmetry¹. In this framework, an off-shell Lagrangian formulation was constructed, with the harmonic superspace approach [12, 13], for a particular class of four-dimensional [3] and three-dimensional [6] $\mathcal{N}=4$ mechanics, with self-dual non-Abelian background. The same idea of coupling with an auxiliary “semi-dynamical” supermultiplet has been also elaborated in [7], within the standard $\mathcal{N}=4$ superspace framework, and then it has been applied for the construction of the Lagrangian and Hamiltonian formulations of the $\mathcal{N}=4$ supersymmetric system describing the motion of the isospin particles in three [8] and four-dimensional [9] conformally flat manifolds, carrying the non-Abelian fields of the Wu-Yang monopole and BPST instanton, respectively.

In both these approaches the additional fermions were completely auxiliary, and they were expressed through the physical ones on the mass shell. Another approach based on the direct use of the $SU(2)$ reduction, firstly considered on the Lagrangian level in the purely bosonic case in [1], has been used in supersymmetric case in [10]. The key idea of this approach is to perform a direct $su(2)$ Hamiltonian reduction in the bosonic sector of the $\mathcal{N}=4$ supersymmetric system, with the general $SU(2)$ invariant action for a self-coupled $(4, 4, 0)$ supermultiplet. No auxiliary superfields are needed within such an approach, and the procedure itself is remarkably simple and automatically successful.

As concerning the interaction with the non-Abelian background, the system considered in [10] was not too illuminating, due to its small number (only one) of physical bosons. In the present Letter we extend the construction of [10] to the case of the $\mathcal{N}=4$ supersymmetric system with five (and four in the special case) physical bosonic components. It is not, therefore, strange that the arising non-Abelian background coincides with the field of a Yang monopole (a BPST instanton field in the four dimensional case). The very important preliminary step, we discussed in details in Section 2, is to pass to new bosonic and fermionic variables, which are inert under the $SU(2)$ group, over which we perform the reduction. Thus, the $SU(2)$ group rotates only the three bosonic components, which enter the action through $SU(2)$ invariant currents. Just these bosonic fields become the “isospin” variables which the background field couples to. Due to the commutativity of $\mathcal{N}=4$ supersymmetry with the reduction $SU(2)$ group, it survives upon reduction. In Section 3 we present the corresponding supercharges and Hamiltonian, which form a standard $\mathcal{N}=4$ superalgebra. In Section 4 we consider some possible generalizations, which include a system with more general bosonic action, a four-dimensional system which still includes eight fermionic components, and the variant of five-dimensional $\mathcal{N}=4$ mechanics constructed with the help of ordinary and twisted $\mathcal{N}=4$ hypermultiplets. Finally, in the Conclusion we discuss some unsolved problems and possible extensions of the present construction.

2 $(8_B, 8_F) \rightarrow (5_B, 8_F)$ reduction and Yang monopole

As the first nontrivial example of the $SU(2)$ reduction in $\mathcal{N}=4$ supersymmetric mechanics we consider the reduction from the eight-dimensional bosonic manifold to the five dimensional one. To start with, let us choose our basic $\mathcal{N}=4$ superfields to be the two quartets of real $\mathcal{N}=4$ superfields $\mathcal{Q}_A^{i\hat{\alpha}}$ (with $i, \hat{\alpha}, A = 1, 2$) defined in the $\mathcal{N}=4$ superspace $\mathbb{R}^{(1|4)} = (t, \theta_{ia})$ and subjected to the constraints

$$D^{(ia}\mathcal{Q}_A^{j)\hat{\alpha}} = 0, \quad \text{and} \quad (\mathcal{Q}_A^{i\hat{\alpha}})^\dagger = \mathcal{Q}_{i\hat{\alpha}A}, \quad (2.1)$$

where the corresponding covariant derivatives have the form

$$D^{ia} = \frac{\partial}{\partial \theta_{ia}} + i\theta^{ia}\partial_t, \quad \text{so that} \quad \{D^{ia}, D^{jb}\} = 2i\epsilon^{ij}\epsilon^{ab}\partial_t. \quad (2.2)$$

¹Note, that the first realization of this idea was proposed in [5]

Each of these $\mathcal{N}=4$ supermultiplets describes four bosonic and four fermionic variables off-shell [13, 14, 15, 16, 17, 18].

The most general action for $\mathcal{Q}_A^{i\hat{\alpha}}$ superfields is constructed by integrating an arbitrary superfunction $\mathcal{F}(\mathcal{Q}_A^{i\hat{\alpha}})$ over the whole $\mathcal{N}=4$ superspace. Here, we restrict ourselves to the simplest prepotential of the form²

$$\mathcal{F}(\mathcal{Q}_A^{i\hat{\alpha}}) = \mathcal{Q}_A^{i\hat{\alpha}} \mathcal{Q}_{i\hat{\alpha}A} \quad \longrightarrow \quad S = \int dt d^4\theta \mathcal{Q}_A^{i\hat{\alpha}} \mathcal{Q}_{i\hat{\alpha}A}. \quad (2.3)$$

The rationale for this selection is, first of all, its manifest invariance under $su(2)$ transformations acting on the “ $\hat{\alpha}$ ” index of $\mathcal{Q}^{i\hat{\alpha}}$. This is the symmetry over which we are going to perform the $su(2)$ reduction. Secondly, just this form of the prepotential guarantees $SO(5)$ symmetry in the bosonic sector after reduction.

In terms of components the action (2.3) reads

$$S = \int dt \left[\dot{Q}_A^{i\hat{\alpha}} \dot{Q}_{i\hat{\alpha}A} - \frac{i}{8} \dot{\Psi}_A^{a\hat{\alpha}} \Psi_{a\hat{\alpha}A} \right] \quad (2.4)$$

where the bosonic and fermionic components are defined as

$$Q_A^{i\hat{\alpha}} = \mathcal{Q}_A^{i\hat{\alpha}}|, \quad \Psi_A^{a\hat{\alpha}} = D^{ia} \mathcal{Q}_{iA}^{\hat{\alpha}}|, \quad (2.5)$$

and, as usually, $(\dots)|$ denotes the $\theta_{ia} = 0$ limit. Thus, from beginning we have just the sum of two independent non-interacting $(4, 4, 0)$ supermultiplets.

To proceed further we define the following bosonic $q_A^{i\alpha}$ and fermionic $\psi_A^{a\alpha}$ fields

$$q_A^{i\alpha} \equiv Q_{A\hat{\alpha}}^i G^{\alpha\hat{\alpha}}, \quad \psi_A^{a\alpha} \equiv \Psi_{\hat{\alpha}A}^a G^{\alpha\hat{\alpha}}, \quad (2.6)$$

where the bosonic variables $G^{\alpha\hat{\alpha}}$, subjected to $G^{\alpha\hat{\alpha}} G_{\alpha\hat{\alpha}} = 2$, are chosen as

$$G^{11} = \frac{e^{-\frac{i}{2}\phi}}{\sqrt{1+\Lambda\bar{\Lambda}}} \Lambda, \quad G^{21} = -\frac{e^{-\frac{i}{2}\phi}}{\sqrt{1+\Lambda\bar{\Lambda}}}, \quad G^{22} = (G^{11})^\dagger, \quad G^{12} = -(G^{21})^\dagger. \quad (2.7)$$

The variables $G^{\alpha\hat{\alpha}}$ play the role of bridge relating two different $SU(2)$ groups realized on the indices α and $\hat{\alpha}$ respectively. Thus, we split our eight bosonic variables $Q_{A\hat{\alpha}}^i$ (2.5) into five ones in $q_A^{i\alpha}$ and three fields $(\phi, \Lambda, \bar{\Lambda})$ entering via $G^{\alpha\hat{\alpha}}$ (2.7).

Now our action (2.4) acquires the form

$$S = \int dt \left[\dot{q}_A^{i\alpha} \dot{q}_{i\alpha A} - 2q_A^{i\alpha} \dot{q}_{iA}^\beta J_{\alpha\beta} + \frac{q_A^{i\alpha} q_{i\alpha A}}{2} J^{\beta\gamma} J_{\beta\gamma} - \frac{i}{8} \dot{\psi}_A^{a\alpha} \psi_{a\alpha A} + \frac{i}{8} \psi_A^{a\alpha} \dot{\psi}_{aA}^\beta J_{\alpha\beta} \right], \quad (2.8)$$

where

$$J^{\alpha\beta} = J^{\beta\alpha} = G^{\alpha\hat{\alpha}} \dot{G}_{\hat{\alpha}}^\beta. \quad (2.9)$$

The new variables $q_A^{i\alpha}$ and $\psi_A^{a\alpha}$, which, clearly, contain five independent bosonic and eight fermionic components, are inert under $su(2)$ rotations of the hatted indices. Under these $su(2)$ rotations, realized now only on $G^{\alpha\hat{\alpha}}$ variables in a standard way

$$\delta G^{\alpha\hat{\alpha}} = \gamma^{(\hat{\alpha}\hat{\beta})} G_{\hat{\beta}}^\alpha,$$

the fields $(\phi, \Lambda, \bar{\Lambda})$ (2.7) transform as [18]

$$\delta\Lambda = \gamma^{11} e^{i\phi} (1+\Lambda\bar{\Lambda}), \quad \delta\bar{\Lambda} = \gamma^{22} e^{-i\phi} (1+\Lambda\bar{\Lambda}), \quad \delta\phi = -2i\gamma^{12} + i\gamma^{22} e^{-i\phi} \Lambda - i\gamma^{11} e^{i\phi} \bar{\Lambda}. \quad (2.10)$$

It is easy to check that the forms $J^{\alpha\beta}$ (2.9) entering the action (2.8) and having the following explicit form

$$J^{11} = -\frac{\dot{\Lambda} - i\Lambda\dot{\phi}}{1+\Lambda\bar{\Lambda}}, \quad J^{22} = -\frac{\dot{\bar{\Lambda}} + i\bar{\Lambda}\dot{\phi}}{1+\Lambda\bar{\Lambda}} \quad \text{and} \quad J^{12} = -i\frac{1-\Lambda\bar{\Lambda}}{1+\Lambda\bar{\Lambda}}\dot{\phi} - \frac{\dot{\Lambda}\bar{\Lambda} - \Lambda\dot{\bar{\Lambda}}}{1+\Lambda\bar{\Lambda}} \quad (2.11)$$

²We used the following definition of the superspace measure: $d^4\theta \equiv -\frac{1}{96} D^{ia} D_{ib} D^{bj} D_{ja}$.

are also invariant under (2.10), as is the whole action (2.8).

Next, we introduce the standard Poisson brackets

$$\{\pi, \Lambda\} = 1, \quad \{\bar{\pi}, \bar{\Lambda}\} = 1, \quad \{p_\phi, \phi\} = 1, \quad (2.12)$$

so that the generators of the transformations (2.10),

$$I_\phi = p_\phi, \quad I = e^{i\phi}[(1+\Lambda\bar{\Lambda})\pi - i\bar{\Lambda}p_\phi], \quad \bar{I} = e^{-i\phi}[(1+\Lambda\bar{\Lambda})\bar{\pi} + i\Lambda p_\phi], \quad (2.13)$$

will be the Noether constants of motion for the action (2.8). To perform the reduction over this SU(2) group we fix the Noether constants as (c.f. [1])

$$I_\phi = m \quad \text{and} \quad I = \bar{I} = 0, \quad (2.14)$$

which yields

$$p_\phi = m \quad \text{and} \quad \pi = \frac{im\bar{\Lambda}}{1+\Lambda\bar{\Lambda}}, \quad \bar{\pi} = -\frac{im\Lambda}{1+\Lambda\bar{\Lambda}}. \quad (2.15)$$

Performing a Routh transformation over the variables $(\Lambda, \bar{\Lambda}, \phi)$, we reduce the action (2.8) to

$$\tilde{S} = S - \int dt \{ \pi \dot{\Lambda} + \bar{\pi} \dot{\bar{\Lambda}} + p_\phi \dot{\phi} \} \quad (2.16)$$

and substitute the expressions (2.15) into \tilde{S} .

As the final step we have to choose the proper parametrization for bosonic components $q_A^{i\alpha}$ (2.6) remembering that they contain only five independent variables. Following [1] we will choose these variables as

$$q_1^{i\alpha} = \frac{1}{2}\epsilon^{i\alpha}\sqrt{r+z_5}, \quad q_2^{i\alpha} = \frac{1}{\sqrt{2(r+z_5)}}\left(x^{(i\alpha)} - \frac{1}{\sqrt{2}}\epsilon^{i\alpha}z_4\right), \quad (2.17)$$

where

$$x^{12} = \frac{i}{\sqrt{2}}z_3, \quad x^{11} = \frac{1}{\sqrt{2}}(z_1 + iz_2), \quad x^{22} = \frac{1}{\sqrt{2}}(z_1 - iz_2), \quad \text{and} \quad r^2 = \sum_{i=1}^5 z_i z_i, \quad (2.18)$$

and five independent fields are $z_m, m = 1, \dots, 5$. A slightly lengthy but straightforward calculations give the action

$$S_{\text{red}} = \int dt \left[\frac{1}{4r} \dot{z}_m \dot{z}_m - \frac{i}{8} \dot{\psi}_A^{a\alpha} \psi_{a\alpha A} + \frac{i}{4r} H^{\alpha\beta} V_{\alpha\beta} + \frac{1}{128r} H^{\alpha\beta} H_{\alpha\beta} - \frac{m^2}{r} - \frac{m}{4r} v^\alpha \bar{v}^\beta H_{\alpha\beta} - \frac{4im}{r} v^\alpha \bar{v}^\beta V_{\alpha\beta} + im (v^\alpha \dot{\bar{v}}_\alpha - v^\alpha \dot{\bar{v}}_\alpha) \right]. \quad (2.19)$$

Here

$$H^{\alpha\beta} = \psi_A^{a\alpha} \psi_{aA}^\beta, \quad v^\alpha = G^{\alpha 1}, \quad \bar{v}^\alpha = G^{\alpha 2}, \quad v^\alpha \bar{v}_\alpha = 1, \quad (2.20)$$

and to ensure that the reduction constraints (2.15) are satisfied we added Lagrange multiplier terms (last two terms in (2.19)). Finally, the variables $V^{\alpha\beta}$ entering the action (2.19) are defined in a rather symmetric way to be

$$V^{\alpha\beta} = \frac{1}{2} \left(q_A^{i\alpha} \dot{q}_{iA}^\beta + q_A^{i\beta} \dot{q}_{iA}^\alpha \right). \quad (2.21)$$

To clarify the relations of these variables with the potential of Yang monopole, one has to introduce the following isospin currents (which will form $su(2)$ algebra upon quantization):

$$T^I = v^\alpha (\sigma^I)_\alpha^\beta \bar{v}_\beta, \quad I = 1, 2, 3. \quad (2.22)$$

Now, the ‘‘harmonics’’ ($v^\alpha \bar{v}^\beta$)-dependent terms in the action (2.19) can be rewritten as

$$-\frac{m}{4r} v^\alpha \bar{v}^\beta H_{\alpha\beta} - \frac{4im}{r} v^\alpha \bar{v}^\beta V_{\alpha\beta} = m T^I \left(\frac{1}{8r} H^I + \frac{1}{r(r+z_5)} \eta_{\mu\nu}^I z_\mu \dot{z}_\nu \right), \quad \mu, \nu = 1, 2, 3, 4. \quad (2.23)$$

Here, we defined the fermionic spin currents

$$H^I = H_\beta^\alpha (\sigma^I)_\alpha^\beta \quad (2.24)$$

and a self-dual t'Hooft symbol

$$\eta_{\mu\nu}^I = \delta_\mu^I \delta_{\nu 4} - \delta_\nu^I \delta_{\mu 4} + \epsilon_{\mu\nu 4}^A. \quad (2.25)$$

Thus we conclude, that the action (2.19) describes $\mathcal{N}=4$ supersymmetric five-dimensional isospin particles moving in the field of Yang monopole

$$\mathcal{A}_\mu = -\frac{1}{r(r+z_5)} \eta_{\mu\nu}^I z_\nu T^I. \quad (2.26)$$

We stress that the $su(2)$ reduction algebra, realized in (2.10), commutes with all (super)symmetries of the action (2.4). Therefore, all symmetry properties of the theory are preserved in our reduction and the final action (2.19) possesses $\mathcal{N}=4$ supersymmetry.

3 Hamiltonian and Supercharges

Within our approach the construction of the supercharges is straightforward. To simplify the expressions for the supercharges in this Section we will choose a slightly different parametrization of the physical components as compared to (2.17). Namely, we will choose the physical bosonic components as follows:

$$q_1^{i\alpha} = \frac{1}{\sqrt{2}} \epsilon^{i\alpha} e^{\frac{1}{2}u}, \quad q_2^{i\alpha}. \quad (3.1)$$

With this parametrization the reduced action (2.19) reads

$$\begin{aligned} S_{\text{red}} = & \int dt \left[\frac{1}{4} e^u \dot{u}^2 + \frac{e^u}{e^u + y} (\dot{q}_2 \cdot \dot{q}_2) + \frac{1}{e^u + y} (q_2 \cdot \dot{q}_2)^2 + \frac{1}{128(e^u + y)} (H \cdot H) + \frac{i}{4(e^u + y)} (V \cdot H) \right. \\ & - \frac{m^2}{e^u + y} - \frac{m}{4(e^u + y)} v^\alpha \bar{v}^\beta H_{\alpha\beta} - \frac{4im}{e^u + y} v^\alpha \bar{v}^\beta V_{\alpha\beta} \\ & \left. - \frac{i}{8} (\dot{\psi}_1 \cdot \psi_1 + \dot{\psi}_2 \cdot \psi_2) + im (\dot{v}^\alpha \bar{v}_\alpha - v^\alpha \dot{\bar{v}}_\alpha) \right]. \end{aligned} \quad (3.2)$$

Here,

$$y = q_2^{i\alpha} q_{2\ i\alpha}, \quad (3.3)$$

while the currents $J^{\alpha\beta}, V^{\alpha\beta}, H^{\alpha\beta}$ and the “harmonics” v^α, \bar{v}_α are defined as previously in (2.9), (2.20) and (2.21).

One may check that the action (3.2) is invariant under $\mathcal{N}=4$ supersymmetry transformations (with parameters μ^{ia})

$$\begin{aligned} \delta u &= -\frac{1}{\sqrt{2}} e^{-\frac{1}{2}u} \mu^{ia} \psi_{1\ i\alpha}, \quad \delta G^{i\hat{\alpha}} = \frac{1}{2\sqrt{2}} e^{-\frac{1}{2}u} (\mu^{jb} \psi_{1\ b\hat{j}} G^{i\hat{\alpha}} - 2\mu^{ib} \psi_{1\ b\hat{j}} G^{j\hat{\alpha}}), \\ \delta \psi_1^{a\alpha} &= -i\sqrt{2} \mu^{a\alpha} e^{\frac{1}{2}u} \dot{u} - i2\sqrt{2} e^{\frac{1}{2}u} \mu^{\beta a} J_{\beta}{}^{\alpha} + \frac{e^{-\frac{1}{2}u}}{2\sqrt{2}} (\mu^{jb} \psi_{1\ b\hat{j}} \psi_1^{a\alpha} - 2\mu^{\alpha b} \psi_{1\ b\hat{j}} \psi_1^{aj}), \\ \delta q_2^{i\alpha} &= \frac{1}{2} \mu^{ia} \psi_{2\ a}{}^{\alpha} + \frac{1}{2\sqrt{2}} e^{-\frac{1}{2}u} (\mu^{jb} \psi_{1\ b\hat{j}} q_2^{i\alpha} - 2\mu^{\alpha b} \psi_{1\ b\hat{j}} q_2^{i\beta}), \\ \delta \psi_2^{a\alpha} &= 4i\mu_j{}^a \dot{q}_2^{j\alpha} + 4i\mu_j{}^a q_2^{j\beta} J_{\beta}{}^{\alpha} + \frac{1}{2\sqrt{2}} e^{-\frac{1}{2}u} (\mu^{jb} \psi_{1\ b\hat{j}} \psi_2^{a\alpha} - 2\mu^{\alpha b} \psi_{1\ b\hat{j}} \psi_2^{a\beta}) \end{aligned} \quad (3.4)$$

if we take into account that on the reduction constraints (2.15) the current $J^{\alpha\beta}$ (2.9) acquires form

$$J^{\alpha\beta} = \frac{2i}{e^u + y} \left(mK^{\alpha\beta} - \frac{1}{16} H^{\alpha\beta} - iV^{\alpha\beta} \right), \quad (3.5)$$

with

$$K^{\alpha\beta} = \frac{1}{2} (v^\alpha \bar{v}^\beta + v^\beta \bar{v}^\alpha). \quad (3.6)$$

To construct the Hamiltonian, as usual, one should define the momenta $(p_u, p_{i\alpha}, \pi_{1\alpha\alpha}, \pi_{2\alpha\alpha}, p_\alpha, \bar{p}^\alpha)$ for the variables $(u, q_2^{i\alpha}, \psi_1^{a\alpha}, \psi_2^{a\alpha}, v^\alpha, \bar{v}_\alpha)$, respectively

$$\begin{aligned} p_u &= \frac{1}{2}e^u \dot{u}, \quad p_{i\alpha} = \frac{2}{e^u + y} (e^u \dot{q}_{2\ i\alpha} + (q_2 \cdot \dot{q}_2) q_{2\ i\alpha}) + \frac{4i}{e^u + y} \left(mK_{\alpha\beta} - \frac{1}{16}H_{\alpha\beta} \right) q_{2\ i}^\beta, \\ \pi_{1\alpha\alpha} &= \frac{i}{8}\psi_{1\alpha\alpha}, \quad \pi_{2\alpha\alpha} = \frac{i}{8}\psi_{2\alpha\alpha}, \quad p_\alpha = im\bar{v}_\alpha, \quad \bar{p}^\alpha = -imv^\alpha. \end{aligned} \quad (3.7)$$

Now we introduce the canonical Poisson brackets

$$\begin{aligned} \{u, p_u\} &= 1, \quad \{q_2^{i\alpha}, p_{j\beta}\} = \delta_j^i \delta_\beta^\alpha, \quad \{v^\alpha, p_\beta\} = -\delta_\beta^\alpha, \quad \{\bar{v}_\alpha, \bar{p}^\beta\} = -\delta_\alpha^\beta, \\ \{\psi_1^{a\alpha}, \pi_{1\ b\beta}\} &= -\delta_b^a \delta_\beta^\alpha, \quad \{\psi_2^{a\alpha}, \pi_{2\ b\beta}\} = -\delta_b^a \delta_\beta^\alpha. \end{aligned} \quad (3.8)$$

From the explicit form of the fermionic $(\pi_{1\alpha\alpha}, \pi_{2\alpha\alpha})$ and bosonic $(p_\alpha, \bar{p}^\alpha)$ momenta (3.8) it follows that we have a second-class constraints. In order to resolve them, one has pass to the Dirac brackets for the canonical variables³

$$\begin{aligned} \{u, p_u\} &= 1, \quad \{q_2^{i\alpha}, p_{j\beta}\} = \delta_j^i \delta_\beta^\alpha, \quad \{v^\alpha, \bar{v}_\beta\} = \frac{i}{2m} \delta_\beta^\alpha, \\ \{\psi_1^{a\alpha}, \psi_{1\ b\beta}\} &= 4i\delta_b^a \delta_\beta^\alpha, \quad \{\psi_2^{a\alpha}, \psi_{2\ b\beta}\} = 4i\delta_b^a \delta_\beta^\alpha. \end{aligned} \quad (3.9)$$

Let us note, that in virtue of (3.9) the currents $K_{\alpha\beta}$ (3.6) obey to $su(2)$ algebra

$$\{K^{\alpha\beta}, K^{\gamma\rho}\} = -\frac{i}{4m} (\epsilon^{\alpha\gamma} K^{\beta\rho} + \epsilon^{\beta\rho} K^{\alpha\gamma} + \epsilon^{\alpha\rho} K^{\beta\gamma} + \epsilon^{\beta\gamma} K^{\alpha\rho}). \quad (3.10)$$

Finally, one may check that the following supercharges

$$\begin{aligned} \mathbb{Q}_{ia} &= \frac{1}{\sqrt{2}} e^{-\frac{1}{2}u} p_u \psi_{1\ ai} - \frac{1}{2} p_{i\alpha} \psi_{2\ a}^\alpha - \frac{1}{2\sqrt{2}} ((q_2 \cdot p) \psi_{1\ ai} + 2\psi_{1\ a\gamma} q_{2\ k}^\gamma p^k_i) \\ &\quad + i\sqrt{2} e^{-\frac{1}{2}u} \left(mK_{ij} - \frac{1}{16}H_{ij} \right) \psi_{1\ a}^j \end{aligned} \quad (3.11)$$

and the Hamiltonian⁴

$$\mathbb{H} = e^{-u} p_u^2 + \frac{(q_2 \cdot p)^2}{4(e^u + y)} + \frac{e^{-u}}{4(e^u + y)} \mathcal{P}^{i\alpha} \mathcal{P}_{i\alpha} - \frac{2}{e^u + y} \left(mK_{\alpha\beta} - \frac{1}{16}H_{\alpha\beta} \right) \left(mK^{\alpha\beta} - \frac{1}{16}H^{\alpha\beta} \right), \quad (3.12)$$

where

$$\mathcal{P}_{i\alpha} = (e^u + y) p_{i\alpha} - q_{2\ i\alpha} (q_2 \cdot p) - 4i \left(mK_{\alpha\beta} - \frac{1}{16}H_{\alpha\beta} \right) q_{2\ i}^\beta, \quad (3.13)$$

form the standard $\mathcal{N}=4$ superalgebra

$$\{\mathbb{Q}^{ia}, \mathbb{Q}_{jb}\} = 2i\delta_j^i \delta_b^a \mathbb{H}. \quad (3.14)$$

It is interesting to note that the term with the ‘‘coupling constant’’ m , which defines the interaction with the Yang monopole, enters the supercharges \mathbb{Q}_{ia} (3.11) in a simple way similar to [19, 20].

Another funny peculiarity of the supercharges (3.11) and Hamiltonian (3.12) is their invariance with respect to simultaneous ‘‘reflection’’ $(q_2, p, \psi_2) \rightarrow (-q_2, -p, -\psi_2)$. This invariance means that one may immediately go to the simplest case by consistently putting $(q_2, p, \psi_2) = 0$. The resulting supercharges and Hamiltonian will describe the one-dimensional system discussed in [5, 7, 10].

With this, we completed the classical description of $\mathcal{N}=4$ five-dimensional supersymmetric mechanics describing the isospin particle interacting with a Yang monopole. Next, we analyze some possible extensions of the present system, together with some possible interesting special cases.

³From now on, the symbol $\{, \}$ stands for the Dirac brackets.

⁴Note, that in virtue of (3.6) and (2.20) $K^{\alpha\beta} K_{\alpha\beta} = -1/2$.

4 Generalizations and the cases of a special interest

In the previous Sections we have analyzed the simplest variant of $SU(2)$ reduction procedure applied to the free eight-dimensional system with $\mathcal{N}=4$ supersymmetry. Here we will consider its possible generalizations concentrating on the bosonic sector only, while the full supersymmetric action could be easily reconstructed, if needed.

4.1 $SO(4)$ invariant systems

The most general system which still possesses $SO(4)$ symmetry upon $SU(2)$ reduction is specified by the prepotential \mathcal{F} (2.3) depending on two scalars X and Y

$$\mathcal{F} = \mathcal{F}(X, Y), \quad X = \mathcal{Q}_1^{i\hat{\alpha}} \mathcal{Q}_{1\ i\hat{\alpha}}, \quad Y = \mathcal{Q}_2^{i\hat{\alpha}} \mathcal{Q}_{2\ i\hat{\alpha}}. \quad (4.1)$$

Such a system is invariant under $SU(2)$ transformations realized on the “hatted” indices $\hat{\alpha}$ and thus the $SU(2)$ reduction we discussed in the Section 2 goes in the same manner. In addition the full $SU(2) \times SU(2)$ symmetry realized on the superfield $\mathcal{Q}_2^{i\hat{\alpha}}$ will survive in the reduction process. So we expected the final system will possess $SO(4)$ symmetry.

The bosonic sector of the system with prepotential (4.1) is described by the action

$$S = \int dt \left[\left(F_x + \frac{1}{2} x F_{xx} \right) \dot{Q}_1^{i\hat{\alpha}} \dot{Q}_{1\ i\hat{\alpha}} + \left(F_y + \frac{1}{2} y F_{yy} \right) \dot{Q}_2^{i\hat{\alpha}} \dot{Q}_{2\ i\hat{\alpha}} + 2 F_{xy} \mathcal{Q}_2^{j\hat{\beta}} \mathcal{Q}_{1\ j\hat{\alpha}} \dot{Q}_{2\ i\hat{\beta}} \dot{Q}_1^{i\hat{\alpha}} \right]. \quad (4.2)$$

Even with a such simple prepotential the bosonic action (4.2) after reduction has a rather complicated form. Next, still meaningful simplification, could be achieved with the following prepotential

$$\mathcal{F} = \mathcal{F}(X, Y) = \mathcal{F}_1(X) + \mathcal{F}_2(Y), \quad (4.3)$$

where $\mathcal{F}_1(X)$ and $\mathcal{F}_2(Y)$ are arbitrary functions depending on X and Y , respectively. With a such prepotential the third term in the action (4.2) disappeared and the action acquires readable form. With our notations (2.17), (2.18) the reduced action reads

$$\begin{aligned} S = \int dt & \left[\frac{H_x H_y}{2((H_x - H_y)z_5 + (H_x + H_y)r)} \dot{z}_\mu \dot{z}_\mu + \frac{(H_x - H_y)^2}{8r^2((H_x - H_y)z_5 + (H_x + H_y)r)} (z_\mu \dot{z}_\mu)^2 + \right. \\ & + \frac{H_x - H_y}{4r^2} (z_\mu \dot{z}_\mu) \dot{z}_5 + \frac{1}{8} \left(\frac{H_x - H_y}{r^2} z_5 + \frac{H_x + H_y}{r} \right) \dot{z}_5^2 + im(\dot{v}^\alpha \bar{v}_\alpha - v^\alpha \dot{\bar{v}}_\alpha) \\ & \left. - \frac{2m^2}{(H_x + H_y)r + (H_x - H_y)z_5} - \frac{8imH_y}{(H_x + H_y)r + (H_x - H_y)z_5} v^\alpha \bar{v}^\beta V_{\alpha\beta} \right], \end{aligned} \quad (4.4)$$

where

$$H_x = F_1'(x) + \frac{1}{2} x F_1''(x), \quad H_y = F_2'(y) + \frac{1}{2} y F_2''(y), \quad (4.5)$$

and

$$x = \frac{1}{2}(r + z_5), \quad y = \frac{1}{2}(r - z_5). \quad (4.6)$$

Let us stress, that the unique possibility to have $SO(5)$ invariant bosonic sector is to choose $H_x = H_y = \text{const}$. This is just the case we considered in the Section 2. With arbitrary potentials H_x and H_y we have a more general system with the action (4.4), describing the motion of the $\mathcal{N}=4$ supersymmetric particle in five dimensions and interacting with Yang monopole and some specific potential.

4.2 Non-linear supermultiplet

It is known for a long time that in some special cases one could reduce the action for hypermultiplets to the action containing one less physical bosonic components – to the action of so-called non-linear supermultiplet [13, 18, 21]. The key idea of such reduction is replacement of the time derivative of the “radial” bosonic component of hypermultiplet $\text{Log}(q^{ia} q_{ia})$ by an auxiliary component B without breaking of $\mathcal{N}=4$ supersymmetry [22]:

$$\frac{d}{dt} \text{Log}(q^{ia} q_{ia}) \rightarrow B. \quad (4.7)$$

Clearly, to perform such replacement in some action the “radial” bosonic component has to enter this action only with time derivative. This condition is strictly constraints the variety of the possible hypermultiplet actions in which this reduction works.

To perform the reduction from hypermultiplet to the non-linear one, the parametrization (2.17) is not very useful. Instead, we choose the following parameterizations for independent components of two hypermultiplets $q_1^{i\alpha}$ and $q_2^{i\alpha}$

$$q_1^{i\alpha} = \frac{1}{\sqrt{2}}\epsilon^{i\alpha}e^{\frac{1}{2}u}, \quad q_2^{i\alpha} = x^{(i\alpha)} - \frac{1}{\sqrt{2}}\epsilon^{i\alpha}z_4, \quad (4.8)$$

where, as before (2.18)

$$x^{12} = \frac{i}{\sqrt{2}}z_3, \quad x^{11} = \frac{1}{\sqrt{2}}(z_1 + iz_2), \quad x^{22} = \frac{1}{\sqrt{2}}(z_1 - iz_2). \quad (4.9)$$

Thus, the five independent components are u and $z_\mu, \mu = 1, \dots, 4$, and

$$x = q_1^2 = e^u, \quad y = q_2^2 = \sum_{\mu=1}^4 z_\mu z_\mu \equiv r_4^2. \quad (4.10)$$

With this parametrization the action (4.4) acquires the form

$$S = \int dt \left[\frac{G_1 G_2 e^u}{e^u G_1 + G_2 r_4^2} \dot{z}_\mu \dot{z}_\mu + \frac{G_2^2}{e^u G_1 + G_2 r_4^2} (z_\mu \dot{z}_\mu)^2 + \frac{1}{4} G_1 e^u \dot{u}^2 + im(\dot{v}^\alpha \bar{v}_\alpha - v^\alpha \dot{\bar{v}}_\alpha) - \frac{m^2}{e^u G_1 + G_2 r_4^2} - \frac{4imG_2}{e^u G_1 + G_2 r_4^2} v^\alpha \bar{v}^\beta V_{\alpha\beta} \right], \quad (4.11)$$

where

$$G_1 = G_1(u) = F'_1(x) + \frac{1}{2}x F''_1(x), \quad G_2 = G_2(r_4) = F'_2(y) + \frac{1}{2}y F''_2(y). \quad (4.12)$$

From explicit form of the action (4.4) follows, that if we choose $G_1 = e^{-u}$, than the “radial” bosonic component u will enter the action only through kinetic term $\sim \dot{u}^2$. Thus, performing replacement (4.7) and excluding the auxiliary field B by its equation of motion we will finish with the action

$$S = \int dt \left[\frac{G_2}{1 + G_2 r_4^2} (\dot{z}_\mu \dot{z}_\mu + G_2 (z_\mu \dot{z}_\mu)^2) + im(\dot{v}^\alpha \bar{v}_\alpha - v^\alpha \dot{\bar{v}}_\alpha) - \frac{m^2}{1 + G_2 r_4^2} - \frac{4imG_2}{1 + G_2 r_4^2} v^\alpha \bar{v}^\beta V_{\alpha\beta} \right]. \quad (4.13)$$

The action (4.13) describes the motion of an isospin particle on four-manifold with $SO(4)$ isometry carrying the non-Abelian field of a BPST instanton and some special potential.

4.3 Ordinary and twisted hypermultiplets

One more possibility to generalize the results we presented in the previous Section is to consider simultaneously ordinary hypermultiplet $\mathcal{Q}^{j\hat{\alpha}}$ obeying to (2.1) together with twisted hypermultiplet $\mathcal{V}^{a\hat{\alpha}}$ - a quartet of $\mathcal{N}=4$ superfields subjected to constraints [18]

$$D^{i(a}\mathcal{V}^{b)\hat{\alpha}} = 0, \quad \text{and} \quad (\mathcal{V}^{a\hat{\alpha}})^\dagger = \mathcal{V}_{a\hat{\alpha}}. \quad (4.14)$$

The most general system which is explicitly invariant under $SU(2)$ transformations realized on the “hatted” indices is defined, similarly to (4.1), by the superspace action depending on two scalars X, Y

$$S = \int dt d^4\theta \mathcal{F}(X, Y), \quad X = \mathcal{Q}^{i\hat{\alpha}} \mathcal{Q}_{i\hat{\alpha}}, \quad Y = \mathcal{V}^{a\hat{\alpha}} \mathcal{V}_{a\hat{\alpha}}. \quad (4.15)$$

The bosonic sector of the action (4.15) is a rather simple

$$S = \int dt \left[\left(F_x + \frac{1}{2}x F_{xx} \right) \dot{\mathcal{Q}}^{i\hat{\alpha}} \dot{\mathcal{Q}}_{i\hat{\alpha}} - \left(F_y + \frac{1}{2}y F_{yy} \right) \dot{\mathcal{V}}^{a\hat{\alpha}} \dot{\mathcal{V}}_{a\hat{\alpha}} \right]. \quad (4.16)$$

Thus, we see that the term causes most complicated structure of the action with two hypermultiplets, disappeared in the case of ordinary and twisted hypermultiplets. Clearly, the bosonic action after $SU(2)$ reduction will have the same form (4.4), but with

$$H_x = F_x + \frac{1}{2}x F_{xx}, \quad H_y = -\left(F_y + \frac{1}{2}y F_{yy}\right). \quad (4.17)$$

Here $F = F(x, y)$ is still function of two variables x and y . The mostly symmetric situation again corresponds to the choice

$$H_x = H_y \equiv h(x, y) \quad (4.18)$$

with the action

$$S = \int dt \left[\frac{h}{4r} \dot{z}_m \dot{z}_m + im (\dot{v}^\alpha \bar{v}_\alpha - v^\alpha \dot{\bar{v}}_\alpha) - \frac{m^2}{h r} - \frac{4im}{r} v^\alpha \bar{v}^\beta V_{\alpha\beta} \right]. \quad (4.19)$$

Due to the definitions (4.17), (4.18) the metric $h(x, y)$ cannot be chosen to be fully arbitrary. For example, looking for $SO(5)$ invariant model with $h = h(x + y)$ we could find only two solutions ⁵

$$h_1 = \text{const}, \quad h_2 = 1/(x + y)^3. \quad (4.20)$$

Both solutions describe a cone-like geometry in the bosonic sector.

Finally, we would like to point the attention to the fact that with $h(x, y) = \text{const}$ the bosonic sectors of the systems with two hypermultiplets and with one ordinary and one twisted hypermultiplets are coincide. This is just one more justification that “almost free” systems could be supersymmetrized in the different ways.

5 Conclusion

In the present paper, starting with the non-interacting system of two $\mathcal{N}=4$ hypermultiplets, we perform a reduction over the $SU(2)$ group which commutes with supersymmetry. The resulting system describes the motion of an isospin carrying particle on a conformally flat five-dimensional manifold in the non-Abelian field of a Yang monopole and in some scalar potential. The most important step for this construction is passing to new bosonic and fermionic variables, which are inert under the $SU(2)$ group, over which we perform the reduction. Thus, the $SU(2)$ group rotates only three bosonic components, which enter the action through $SU(2)$ invariant currents. Just these bosonic fields become the “isospin” variables, which the background field couples to. Due to the commutativity of $\mathcal{N}=4$ supersymmetry with the reduction $SU(2)$ group, it survives upon reduction. We also presented the corresponding supercharges and Hamiltonian, which form a standard $\mathcal{N}=4$ superalgebra. Some possible generalizations of the action to the cases of systems with a more general bosonic action, a four-dimensional system which still includes eight fermionic components, and a variant of five-dimensional $\mathcal{N}=4$ mechanics constructed with the help of the ordinary and twisted $\mathcal{N}=4$ hypermultiplets were considered. The main preference of the proposed approach is its applicability to any system which possesses $SU(2)$ invariance. If, in addition, this $SU(2)$ commutes with supersymmetry, then the resulting system will be automatically supersymmetric.

One of the interesting peculiarities of the constructed system is a very simple dependence of the supercharges on the “coupling constant”. Another interesting feature is the existence of two different $\mathcal{N}=4$ supersymmetrizations of the same bosonic action firstly proposed in [1].

Among possible direct applications of our construction there are the reduction in the cases of systems with non-linear $\mathcal{N}=4$ supermultiplets [24], systems with more than two (non-linear) hypermultiplets, in the systems with bigger supersymmetry, say for example $\mathcal{N}=8$, etc. However, the most important case, which is still missing within our approach, is the construction of the $\mathcal{N}=4$ supersymmetric particle on the sphere S^5 in the field of a Yang monopole. Unfortunately, the use of standard linear hypermultiplets makes the solution of this task impossible because the resulting bosonic manifolds have a different structure (conical geometry) to include S^5 . Thus, our hopes to construct such a system are related either to less supersymmetric theories (say with $\mathcal{N}=2$ supersymmetry) or to non-linear supermultiplets.

⁵The same metric has been considered in [23].

Acknowledgements

We thank Armen Nersessian and Francesco Toppan for useful discussions. S.K. and A.S. are grateful to the Laboratori Nazionali di Frascati for hospitality. This work was partially supported by the grants RFBF-09-02-01209 and 09-02-91349, by Volkswagen Foundation grant I/84 496 as well as by the ERC Advanced Grant no. 226455, “*Supersymmetry, Quantum Gravity and Gauge Fields*” (*SUPERFIELDS*).

References

- [1] M. Gonzales, Z. Kuznetsova, A. Nersessian, F. Toppan, V. Yeghikyan, *Second Hopf map and supersymmetric mechanics with Yang monopole*, Phys.Rev. D 80 (2009) 025022, [arXiv:0902.2682\[hep-th\]](#).
- [2] M. Konyushikhin, A. Smilga, *Self-duality and supersymmetry*, Phys. Lett. B 689 (2010) 95, [arXiv:0910.5162\[hep-th\]](#).
- [3] E.A. Ivanov, M.A. Konyushikhin, A.V. Smilga, *SQM with non-Abelian self-dual fields: harmonic superspace description*, JHEP 1005 (2010) 033, [arXiv:0912.3289\[hep-th\]](#).
- [4] S. Bellucci, S. Krivonos, A. Sutulin, *Dual multiplets in $N=4$ superconformal mechanics*, [arXiv:1012.2325v2 \[hep-th\]](#).
- [5] S. Fedoruk, E. Ivanov, O. Lechtenfeld, *Supersymmetric Calogero models by gauging*, Phys. Rev. D 79 (2009) 105015, [arXiv:0812.4276\[hep-th\]](#).
- [6] E. Ivanov, M. Konyushikhin, *$N=4$, 3D Supersymmetric Quantum Mechanics in Non-Abelian Monopole Background*, Phys. Rev. D 82 (2010) 085014, [arXiv:1004.4597\[hep-th\]](#).
- [7] S. Bellucci, S. Krivonos, *Potentials in $N=4$ superconformal mechanics*, Phys. Rev. D 80 (2009) 065022, [arXiv:0905.4633\[hep-th\]](#).
- [8] S. Bellucci, S. Krivonos, A. Sutulin, *Three dimensional $N=4$ supersymmetric mechanics with Wu-Yang monopole*, Phys. Rev. D 81 (2010) 105026, [arXiv:0911.3257\[hep-th\]](#).
- [9] S. Krivonos, O. Lechtenfeld, A. Sutulin, *$N=4$ Supersymmetry and the BPST Instanton*, Phys. Rev. D 81 (2010) 085021, [arXiv:1001.2659\[hep-th\]](#).
- [10] S. Krivonos, O. Lechtenfeld, *$SU(2)$ reduction in $N=4$ supersymmetric mechanics*, Phys. Rev. D 80 (2009) 045019, [arXiv:0906.2469\[hep-th\]](#).
- [11] S.C. Zhang, J.P. Hu, *A Four Dimensional Generalization of the Quantum Hall Effect*, Science 294 (2001) 823, [arXiv:cond-mat/0110572](#),
B.A. Bernevig, J.P. Hu, N. Toumbas, S.C. Zhang, *The Eight Dimensional Quantum Hall Effect and the Octonions*, Phys. Rev. Lett. 91 (2203) 236803, [arXiv:cond-mat/0306045](#),
D. Karabali, V.P. Nair, *Quantum Hall Effect in Higher Dimensions*, Nucl. Phys. B 641 (2002) 533, [arXiv:hep-th/0203264](#),
K. Hasebe, *Hyperbolic Supersymmetric Quantum Hall Effect*, Phys. Rev. D78 (2008) 125024, [arXiv:0809.4885\[hep-th\]](#).
- [12] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky, E.S. Sokatchev, *Harmonic Superspace*, Cambridge, UK: Univ.Press. (2001), 306pp.
- [13] E. Ivanov, O. Lechtenfeld, *$N=4$ Supersymmetric Mechanics in Harmonic Superspace*, JHEP 0309 (2003) 073, [arXiv:hep-th/0307111](#).
- [14] R.A. Coles, G. Papadopoulos, *The Geometry of the one-dimensional supersymmetric nonlinear sigma models*, Class. Quant. Grav. 7 (1990) 427.
- [15] G.W. Gibbons, G. Papadopoulos, K.S. Stelle, *HKT and OKT Geometries on Soliton Black Hole Moduli Spaces*, Nucl. Phys. B 508 (1997) 623, [arXiv:hep-th/9706207](#).

- [16] S. Hellerman, J. Polchinski, *Supersymmetric quantum mechanics from light cone quantization*, in: Shifman, M.A. (ed.), *The many faces of the superworld*, [arXiv:hep-th/9908202](#).
- [17] C.M. Hull, *The geometry of supersymmetric quantum mechanics*, [arXiv:hep-th/9910028](#).
- [18] E. Ivanov, S. Krivonos, O. Lechtenfeld, *$N=4$, $d=1$ supermultiplets from nonlinear realizations of $D(2,1;\alpha)$* , Class. Quant. Grav. 21 (2004) 1031, [arXiv:hep-th/0310299](#).
- [19] S. Krivonos, O. Lechtenfeld, *Many-particle mechanics with $D(2,1;\alpha)$ superconformal symmetry*, JHEP 1102 (2011) 042, [arXiv:1012.4639\[hep-th\]](#).
- [20] S. Bellucci, S. Krivonos, A. Sutulin, *$CP(n)$ supersymmetric mechanics in $U(n)$ background gauge fields*, Phys. Rev. D 84 (2011) 065033, [arXiv:1106.2435\[hep-th\]](#).
- [21] S. Bellucci, S. Krivonos, *Geometry of $N=4$, $d=1$ nonlinear supermultiplet*, Phys. Rev. D 74 (2006) 125024, [arXiv:hep-th/0611104](#);
S. Bellucci, S. Krivonos, V. Ohanyan, *$N=4$ Supersymmetric MICZ-Kepler systems on S^3* , Phys. Rev. D 76 (2007) 105023, [arXiv:0706.1469\[hep-th\]](#).
- [22] S. Bellucci, S. Krivonos, A. Marrani, E. Orazi, *"Root" Action for $N=4$ Supersymmetric Mechanics Theories*, Phys. Rev. D 73 (2006) 025011, [arXiv:hep-th/0511249](#).
- [23] E. Ivanov, O. Lechtenfeld, A. Sutulin, *Hierarchy of $N=8$ Mechanics Models*, Nucl. Phys. B 790 (2008) 493, [arXiv:0705.3064\[hep-th\]](#).
- [24] S. Bellucci, S. Krivonos, O. Lechtenfeld, A. Shcherbakov, *Superfield Formulation of Nonlinear $N=4$ Supermultiplets*, Phys. Rev. D 77 (2008) 045026, [arXiv:0710.3832\[hep-th\]](#).